# An RLT Approach for Solving the Binary-Constrained Mixed Linear Complementarity Problem

# An RLT Approach for Solving the Binary-Constrained Mixed Linear Complementarity Problem

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## Abstract

It is well known that the mixed linear complementarity problem can be used to model equilibria in energy markets as well as a host of other engineering and economic problems. The *binary-constrained mixed linear complementarity problem* is a formulation of the mixed linear complementarity problem in which some variables are restricted to be binary. This paper presents a novel approach for solving the binary-constrained mixed linear complementarity problem. First we solve a series of linear optimization problems that enables us to replace some of the complementarity constraints with linear equations. Then we solve an equivalent mixed integer linear programming formulation of the original binary-constrained mixed linear complementarity problem (with a smaller number of complementarity constraints) to guarantee a solution to the problem. Our computational results on a wide range of test problems, including some engineering examples, demonstrate the usefulness and the effectiveness of this novel approach.

**Keywords:** Mixed Linear Complementarity Problems, Mixed integer linear programming, Reformulation Linearization Technique, Electricity Market,

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Linear Programming.

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## 1. Introduction

The discretely-constrained mixed linear complementarity problem (DC-MLCP) is a formulation of the mixed linear complementarity problem (MLCP) in which some of the variables are restricted to be discrete. The DC-MLCP has many practical applications. For example, it includes a realistic model of market equilibrium in electric power markets and for grid based industries [21, 28, 29, 51, 34]. It also models the interaction between several players in a non-cooperative game with binary decisions, such as transportation and facility location models [9] and agriculture and land-use planning [60]. The important benefits of DC-MLCPs are further described in [17], where their usefulness is shown for solving important problems in transportation networks and power system networks with storage. In most applications of DC-MLCP, the discrete variables are binary and represent the on/off status of the players or other entities involved, such as power generators in the power systems context. For this reason, this paper focuses on DC-MLCP with binary variables; we call this problem the binary-constrained mixed linear complementarity problem (BC-MLCP).

More formally, the mathematical formulation of the BC-MLCP is described as follows. Given the vector $q = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and the matrix $A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \in \mathbb{R}^{n \times n}$, find $\begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $n_1 + n_2 = n$, such that:

$$0 \leq q^1 + \begin{pmatrix} A^{11} & A^{12} \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \perp z^1 \geq 0 \tag{1a}$$

$$0 = q^2 + \begin{pmatrix} A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}, \ z^2 \text{ free,} \tag{1b}$$

where some components of the variable $\begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$ are constrained to be binary. $z^1$ and $z^2$ are called the complementarity and the free variables, respectively.

The DC-MLCP has not been studied widely in the literature. Gabriel et al. [28] proposed a mixed linear integer programming (MILP) approach in which the complementarity constraints, as well as the integrality of the discrete variables are relaxed. They showed that their approach was suitable to energy markets. Their approach has also been successfully used to solve the

Nash-Cournot game with application to power markets [29], and to solve the electricity pool pricing [51].

As opposed to the DC-MLCP, the linear complementarity problem (LCP) has attracted a lot of interest due to its wide range of applications [5, 10, 11, 19, 61]. Several solution techniques have been proposed in the literature for the LCP [20, 40, 46, 36, 39, 1, 4]. Most of these techniques (Lemke's algorithm, Projected Gauss Seidel, etc.) are iterative algorithms and they make use of linear algebra to solve or approximately solve the LCP, see [41, 42, 8, 58, 59, 46]. Some of these methods are impressively successful when the matrix $A$ has a particular structure. The classes of positive semi-definite and P matrices are the most interesting of these classes. For the positive semi-definite matrices, the LCP always has a solution provided it is feasible, i.e., the linear constraints are consistent, as shown by Cottle et al. [10].

It has been shown that solving an LCP is equivalent to finding a stationary point of a bilinear programming problem (BPP) [47]. In this case, the LCP has a solution if and only if there is a global solution to the corresponding BPP with optimal value equal to zero. However, the BPP is an $\mathcal{NP}$-hard problem. Therefore using global optimization techniques for the BPP in the case of the LCP will not make the problem less hard. Al-Khayyal [3], Júdice and Faustino [37], and Júdice [38] have solved the LCP using an equivalent quadratic programming formulation.

In this paper, we propose a new solution approach for the BC-MLCP. Our approach consists of two steps. Firstly, we consider the RLT relaxation of the complementarity constraints and solve a series of LPs that enables us to replace some of these constraints with linear equations. We present theoretical results to show that we do not lose any point that satisfies the BC-MLCP constraints. Then, we have a second step which ensures that it always finds a solution that satisfies all the constraints of the BC-MLCP. This is the approach proposed by Sherali et al. [55] for LCPs. It consists of solving an MILP that is equivalent to the BC-MLCP. We validate our proposed solution approach on some 1400 test instances that include well-known engineering applications.

The rest of this paper is organized as follows. In Section 2, we present the different reformulations of the problem that the reader will encounter in this paper. We review the relevant literature in Section 3. The description of our proposed solution approach is presented in Section 4 and the computational experience is discussed in Section 5. Finally some concluding remarks are given in Section 6.

*Notation:.* Let $N = \{1, \ldots, n_1, n_1 + 1, \ldots, n_1 + n_2\} = \{1, \ldots, n\}$, $B = \{i \in N : z_i \text{ is binary}\}$, $N_1 = \{1, \ldots, n_1\}$ and $N_2 = \{n_1 + 1, \ldots, n_1 + n_2\} = N \setminus N_1$.

In this paper, $n_1$ and $n_2$ designate the number of complementarity and free variables, respectively, and $n = n_1 + n_2$ is the total number of variables.

## 2. Problem formulations

In this section we present the different formulations used throughout this paper.

### *2.1. An equivalent reformulation of the BC-MLCP*

Solving the BC-MLCP is equivalent to finding a point $z^1 \in \mathbb{R}^{n_1}$ and $z^2 \in \mathbb{R}^{n_2}$ that satisfies the following system:

$$\begin{aligned}
q^1 + A^{11}z^1 + A^{12}z^2 &\geq 0 && && \text{(2a)}\\
(q^1 + A^{11}z^1 + A^{12}z^2)^T z^1 &= 0 && && \text{(2b)}\\
q^2 + A^{21}z^1 + A^{22}z^2 &= 0 && && \text{(2c)}\\
z^2 &\text{ free} && && \text{(2d)}\\
z^1 &\geq 0 && && \text{(2e)}\\
z_i^1, z_j^2 &\in \{0,1\} && \text{for } i \in B \cap N_1, j \in B \cap N_2 && \text{(2f)}
\end{aligned}$$

*Assumption 1.* All the continuous variables satisfy $0 \leq z_i^1 \leq u_i^1$ with $u_i^1 \in \mathbb{R}_+$ for all $i \in N_1 \setminus B$, and $l_i^2 \leq z_i^2 \leq u_i^2$ with $u_i^2, l_i^2 \in \mathbb{R}$ and $u_i^2 > l_i^2$ for all $i \in N_2 \setminus B$.

Gabriel et al. [28] give motivations for Assumption 1, and propose a method for finding suitable bounds.

**Lemma 1.** *Suppose that Assumption 1 is in force such that $0 \leq z_i^1 \leq u_i^1$ with $u_i^1 \in \mathbb{R}_+$ for all $i \in N_1 \setminus B$, and $l_i^2 \leq z_i^2 \leq u_i^2$ with $u_i^2, l_i^2 \in \mathbb{R}$ and $u_i^2 > l_i^2$ for all $i \in N_2 \setminus B$. Then, without loss of generality, we can assume that $z_i \in [0,1]$ for all $i \in N \setminus B$, in other words, all the solutions of the BC-MLCP satisfy $z_i \in [0,1]$.*

A proof of Lemma 1 is given in Appendix B.

In order to simplify the notation used so far, we set $x = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$, and we let $A_i \in \mathbb{R}^{1\times n}$ be the $i^{th}$ row of the matrix $A$, and $q_i$ the $i^{th}$ element of the vector $q$. We can therefore define the following set $\mathcal{F}$, which is equivalent $\Big($in the sense that a solution $x$ to any optimization problem over the set $\mathcal{F}$ is a solution to the BC-MLCP$\Big)$ to the set of all the solutions of the BC-MLCP considering Assumption 1.

$$\mathcal{F} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{cc} q_i + A_i x \geq 0 & \text{for } i \in N_1 \\ (q_i + A_i x) x_i = 0 & \text{for } i \in N_1 \\ q_i + A_i x = 0 & \text{for } i \in N_2 \\ 0 \leq x_i \leq 1 & \text{for } i \in N \setminus B \\ x_i \in \{0,1\} & \text{for } i \in B \end{array} \right\}.$$

Note that $\mathcal{F}$ does not directly relate to an MLCP as it is not a square system. This is because the constraints $x_i \leq 1$ for all $i \in N \setminus B$ and $x_i \in \{0,1\}$ for all $i \in B$ do not have associated dual variables. The rest of this paper focuses on finding a point $x \in \mathcal{F}$ that is a solution to the BC-MLCP.

### *2.2. An RLT-based relaxation of the set $\mathcal{F}$*

Following the idea of the RLT relaxation [54], we introduce the variables $y_{ij} = x_i x_j$ for $1 \leq i \leq j \leq n$ and relax the quadratic constraint $y_{ij} = x_i x_j$ with the linear inequalities $y_{ij} \geq 0$, $x_i - y_{ij} \geq 0$, $x_j - y_{ij} \geq 0$ and $y_{ij} - x_i - x_j + 1 \geq 0$ for all pairs $i, j$. These inequalities are also known as the McCormick inequalities [44]. We also relax the binary constraints as $0 \leq x_i \leq 1$ and strengthen this relaxation by adding the constraints $y_{ii} = x_i$ for $i \in B$. These constraints are redundant for $\mathcal{F}$ but can prove useful in the relaxation $\tilde{\mathcal{F}}$.

The RLT relaxation of the set $\mathcal{F}$ is given by the set

$$\tilde{\mathcal{F}} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{cc} q_i + A_i x \geq 0 & \text{for } i \in N_1 \\ q_i x_i + \sum_{j=1}^{n} A_{ij} y_{ij} = 0 & \text{for } i \in N_1 \\ q_i + A_i x = 0 & \text{for } i \in N_2 \\ 0 \leq x_i \leq 1 & \text{for } i \in N \\ y_{ii} = x_i & \text{for } i \in B \\ y_{ij} \geq 0 & \text{for } 1 \leq i \leq j \leq n \\ y_{ij} \leq x_i & \text{for } 1 \leq i \leq j \leq n \\ y_{ij} \leq x_j & \text{for } 1 \leq i \leq j \leq n \\ y_{ij} + 1 \geq x_i + x_j & \text{for } 1 \leq i \leq j \leq n \end{array} \right\}.$$

It is important to note that if both $x_i$ and $x_j$ are binary, then the McCormick inequalities are equivalent to $y_{ij} = x_i x_j$. Otherwise, they simply represent valid inequalities or linear over- and under-estimators of the quadratic equation $y_{ij} = x_i x_j$. This is the reason why $\tilde{\mathcal{F}}$ is a relaxation of the set $\mathcal{F}$. In addition, it is clear that for all $x \in \mathcal{F}$, one can find $y \in \mathbb{R}^{\binom{n}{2}}$ such that $(x, y) \in \tilde{\mathcal{F}}$ (take $y \in \mathbb{R}^{\binom{n}{2}}$ such that its components correspond to $y_{ij} = x_i x_j$). This type of relaxation can sometimes be strengthened further by adding some families of RLT-related cutting planes, see for example [6, 16], or by implementing the *Optimization-Based Bound Tightening (OBBT) [32, 50, 43, 57] procedure for global MINLP.*

### *2.3. Sub-problems related to $\tilde{\mathcal{F}}$*

In this paper, we present an LP-based procedure that successively fixes the complementarity constraints $(q_i + A_i x)x_i = 0$ to either $q_i + A_i x = 0$ or $x_i = 0$, for $i \in N_1$. So, for a given index $i_0 \in N_1$, we define the corresponding relaxed sets as:

$$\tilde{\mathcal{F}}_{i_0}^{+} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{cc} q_i + A_i x \geq 0 & \text{for } i \in N_1 \\ q_i x_i + \sum\limits_{j=1}^{n} A_{ij} y_{ij} = 0 & \text{for } i \in N_1, i \neq i_0 \\ q_i + A_i x = 0 & \text{for } i \in N_2 \\ 0 \leq x_i \leq 1 & \text{for } i \in N \\ y_{ii} = x_i & \text{for } i \in B \\ y_{ij} \geq 0 & \text{for } 1 \leq i \leq j \leq n \\ y_{ij} \leq x_i & \text{for } 1 \leq i \leq j \leq n \\ y_{ij} \leq x_j & \text{for } 1 \leq i \leq j \leq n \\ y_{ij} + 1 \geq x_i + x_j & \text{for } 1 \leq i \leq j \leq n \\ q_{i_0} + A_{i_0} x = 0 & \end{array} \right\}$$

and

$$\tilde{\mathcal{F}}_{i_0}^{0} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{cc} q_i + A_i x \geq 0 & \text{for } i \in N_1 \\ q_i x_i + \sum\limits_{j=1}^{n} A_{ij} y_{ij} = 0 & \text{for } i \in N_1, i \neq i_0 \\ q_i + A_i x = 0 & \text{for } i \in N_2 \\ 0 \leq x_i \leq 1 & \text{for } i \in N \\ y_{ii} = x_i & \text{for } i \in B \\ y_{ij} \geq 0 & \text{for } 1 \leq i \leq j \leq n \\ y_{ij} \leq x_i & \text{for } 1 \leq i \leq j \leq n \\ y_{ij} \leq x_j & \text{for } 1 \leq i \leq j \leq n \\ y_{ij} + 1 \geq x_i + x_j & \text{for } 1 \leq i \leq j \leq n \\ x_{i_0} = 0 & \end{array} \right\}.$$

$\tilde{\mathcal{F}}_{i_0}^{+}$ and $\tilde{\mathcal{F}}_{i_0}^{0}$ differ from $\tilde{\mathcal{F}}$ only in that the $i_0^{th}$ equation of the form $q_i x_i + \sum\limits_{j=1}^{n} A_{ij} y_{ij} = 0$ has been replaced by the equations $q_{i_0} + A_{i_0} x = 0$ and $x_{i_0} = 0$, respectively. These two sets will later be used to simplify notation, especially in Theorem 1 and Algorithm 1.

## 3. Literature review

In this section, we discuss some of the existing literature on the DC-MLCP. Then, we will review the RLT method proposed by Sherali et al. [55] for solving the LCP, which will be extended to BC-MLCP in Section 4.2. For more details on LCPs, we refer the reader to [8, 10].

### *3.1. DC-MLCP review*

One of the early works on discretely-constrained complementarity was by Cunnigham and Geelen [13] who considered using rank-symmetric and principally unimodular matrices. A matrix $M$ is defined as rank-symmetric if $rank(M[X,Y]) = rank(M[Y,X])$ for the submatrix of $M[A,B]$ defined by rows from the set $A$ and columns from the set $B$. Principally unimodular matrices have the property that every nonsingular principal submatrix has determinant $= \pm 1$; they generalize the more familiar totally unimodular matrices. Cunnigham and Geelen [13] showed that if the LCP matrix $M$ is integer-valued, rank-symmetric, and principally unimodular, and $q$ is an integral vector, then if the linear complementarity problem with data $q, M$ has a solution, then it has an integer-valued one. Their results extend those of Chandrasekaran et al. [12] where the authors showed that the LCP matrix $M$ is principally unimodular if and only if for every integral vector $q$, all basic solutions of the LCP$(q, M)$ are integral. In [12], the authors provide a peeling algorithm that can find an integer solution to the related LCP for certain classes of LCP matrices $M$. However in some cases no conclusions can be drawn about the solutions.

Earlier work in Pardalos and Nagurney [49] considered the integer LCP and showed that when the variables are bounded, an equivalent integer linear formulation can be used. However these authors solved the unbounded case using enumeration, and thus the current paper is an improvement on this approach.

Gabriel et al. [28, 29] considered a Pareto optimal approach trading off integrality with an MLCP solution. Ruiz et al. [51] subsequently applied that approach to a power market DC-MLCP. More recently, Huppmann and Siddiqui [34] considered an approach to solve binary Nash equilibrium problems for power markets that interprets dual variables to binary constraints using the benefit or loss from deviations as opposed to marginal relaxations. Their work has a more problem-specific focus than this paper. Lastly, Gabriel [30] considered a DC-MCP reformulation using the concept of a median function and applied it to both the traditional and a bounded MCP.

*3.2. The RLT approach for the LCP*

Sherali et al. [55] proposed an enumerative algorithm that solves the LCP exactly. Their approach solves an equivalent MILP reformulation of the LCP based on the RLT relaxation. In fact they showed that the LCP, as formulated in (1) with $n_2 = 0$, is equivalent to the following MILP:

$$\min_{v,w} \quad q^T v + \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} w_{ij} \tag{3a}$$

$$s.t. \tag{3b}$$

$$\sum_{j=1}^{n} A_{kj} w_{ij} + q_k v_i \geq 0 \qquad \forall (i,k) \tag{3c}$$

$$\sum_{j=1}^{n} A_{kj} x_j + q_k \geq \sum_{j=1}^{n} A_{kj} w_{ij} + q_k v_i \qquad \text{for } i,k \in N \tag{3d}$$

$$0 \leq w_{ij} \leq 1 \qquad \text{for } 1 \leq i \leq j \leq n \tag{3e}$$

$$w_{jj} = x_j \qquad \text{for } j \in N \tag{3f}$$

$$v_j \in \{0,1\} \qquad \text{for } j \in N \tag{3g}$$

$$w_{ij} \geq 0 \qquad \text{for } 1 \leq i \leq j \leq n \tag{3h}$$

$$w_{ij} \leq x_i \qquad \text{for } 1 \leq i \leq j \leq n \tag{3i}$$

$$w_{ij} \leq v_j \qquad \text{for } 1 \leq i \leq j \leq n \tag{3j}$$

$$w_{ij} + 1 \geq x_i + v_j \qquad \text{for } 1 \leq i \leq j \leq n. \tag{3k}$$

Note that $v_i = 0$ if $x_i = 0$ and $v_i = 1$ if $x_i > 0$, and the variable $w_{ij}$ represents the quadratic term $v_i x_j$ for all pair $i, j$. The paper [55] proposed a branch-and-bound method for solving problem (3). It is important to mention that such a branch-and-bound algorithm can converge even faster if some valid inequalities are added to tighten this RLT relaxation. Audet et al. [6] and Djeumou-Fomeni et al. [16] propose several families of cutting planes for this purpose.

The first component of the solution approach proposed in the current paper uses an RLT relaxation of the complementarity constraints. However, it differs from the method in [55] in the sense that it uses the traditional RLT relaxation ($y_{ij} = x_i x_j$) without introducing new binary variables. It also solves a series of LPs in order to replace some of the complementarity constraints with linear equations.

*3.3. The disjunctive-constraints approach*

A common approach to deal with complementarity constraints is to recast them as disjunctive constraints [22]. This approach can also be used to solve

BC-MLCP instances. Starting with the reformulation $\mathcal{F}$ defined in Section 2.1, the complementarity relationship can be recast using the so-called Big-M method to yield the following set:

$$\mathcal{F}_M = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{cc} q_i + A_i x \geq 0 & \text{for } i \in N_1 \\ q_i + A_i x \leq M u_i & \text{for } i \in N_1 \\ x_i \leq M(1-u_i) & \text{for } i \in N_1 \\ q_i + A_i x = 0 & \text{for } i \in N_2 \\ 0 \leq x_i \leq 1 & \text{for } i \in N \setminus B \\ x_i \in \{0,1\} & \text{for } i \in B \\ u_i \in \{0,1\} & \text{for } i \in N_1. \end{array} \right\},$$

where the parameter $M$ is a suitably large positive constant and $u$ is a vector of binary variables.

This approach, while easy to implement, has the drawback that the choice of $M$ is problem-specific, and one must either find the constant $M$ analytically, or proceed by trial-and-error approach to obtain it. Choosing $M$ too small may result in $\mathcal{F}_M \neq \mathcal{F}$, and even $\mathcal{F}_M = \emptyset$; on the other hand, choosing $M$ too large may result in ill-conditioning of the problem and increased run times. For these reasons, we believe that proposing an alternative approach is worthwhile. Furthermore we show in Subsection 5.2 that the Big-M approach can fail on some simple instances of BC-MLCP for which our approach is able to find exact solutions.

## 4. Our solution approach for the BC-MLCP

In this section we describe our proposed solution approach for the BC-MLCP. This approach proceeds in two main steps. In the first step, we solve a series of LPs that enables us to fix some of the complementarity constraints to linear equations. Then, in the second step, we solve an MILP that is equivalent to the original BC-MLCP with possibly a reduced number of complementarity constraints.

### *4.1. Replacement of the complementarity constraints*

In this section we show how we can replace some of the complementarity constraints by solving a series of LPs. One way of finding a point that belongs to the set $\mathcal{F}$ is by solving any optimization problem over the set defined by $\mathcal{F}$. Any such problem will be an $\mathcal{NP}$-hard problem as it is a nonconvex, mixed-integer nonlinear programming problem. However, we show that one can solve a finite series of LPs over the set $\tilde{\mathcal{F}}$ and use the results to replace some of the complementarity constraints with linear equations. In order to do so, we proceed as follows.

For $i_0 \in N_1$, we define the following LP : $\min_{(x,y)\in\tilde{\mathcal{F}}} x_{i_0}$. We can use the solution of this LP to replace the complementarity constraint $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ with a linear equation using the following result.

**Theorem 1.** *Let us assume that $\tilde{\mathcal{F}}$ is non-empty. For $i_0 \in N_1$, let $x^*_{i_0}$ be the optimal objective value of $\min\limits_{(x,y)\in\tilde{\mathcal{F}}} x_{i_0}$. We have the following:*

1. *If $x^*_{i_0} \neq 0$, then all the feasible points of $\tilde{\mathcal{F}}$ are contained in the set defined by $\tilde{\mathcal{F}}^+_{i_0}$. Thus the constraint $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ can be replaced by $q_{i_0} + A_{i_0}x = 0$.*
2. *If $x^*_{i_0} = 0$ and the problem $\min\limits_{(x,y)\in\tilde{\mathcal{F}}^+_{i_0}} x_{i_0}$ is infeasible, then all the feasible points of $\tilde{\mathcal{F}}$ are contained in the set defined by $\tilde{\mathcal{F}}^0_{i_0}$. Thus the constraint $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ can be replaced by $x_{i_0} = 0$.*

*Proof.*

1. By definition the set $\tilde{\mathcal{F}}$ contains the set $\mathcal{F}$. In addition, $x^*_{i_0} \neq 0$ means that there is no feasible point $(x, y) \in \tilde{\mathcal{F}}$ that satisfies $x_{i_0} = 0$. Therefore, there is no such point in $\mathcal{F}$ either, and the complementarity constraint $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ can only be satisfied if $q_{i_0} + A_{i_0}x = 0$.
2. Let the problem $\min\limits_{(x,y)\in\tilde{\mathcal{F}}^+_{i_0}} x_{i_0}$ be infeasible. This means that there is no feasible point $(x, y) \in \tilde{\mathcal{F}}$ that satisfies $q_{i_0} + A_{i_0}x = 0$. Therefore, there is no such point in $\mathcal{F}$ either, and the complementarity constraint $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ can only be satisfied if $x_{i_0} = 0$.

$\square$

*Remarks.*

- It is important to note that when $x^*_{i_0} = 0$ and the problem $\min\limits_{(x,y)\in\tilde{\mathcal{F}}^+_{i_0}} x_{i_0}$ is feasible, no conclusion can be drawn, i.e., we do not have any justifiable reason to replace the complementarity constraint $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ with either $q_{i_0} + A_{i_0}x = 0$ or $x_{i_0} = 0$.

- One can also see the results of Theorem 1 as a partial application of OBBT combined with strong branching on $x_{i_0} = 0 \vee q_{i_0} + A_{i_0}x = 0$. **Indeed, OBBT would normally solve both** $\min_{(x,y)\in\tilde{\mathcal{F}}} x_{i_0}$ **and**

$\max_{(x,y)\in\tilde{\mathcal{F}}} x_{i_0}$**, and use the results to refine the bounds on the variables, whereas Theorem 1 only considers** $\min_{(x,y)\in\tilde{\mathcal{F}}} x_{i_0}$ **and uses the results to possibly replace the nonlinear constraint** $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ **with either** $q_{i_0} + A_{i_0}x = 0$ **or** $x_{i_0} = 0$**.**

- The second part of Theorem 1 can be strengthened if $i_0 \in B$, in which case one could fix $x_{i_0} = 1$ when checking for the infeasibility of $\min_{(x,y)\in\tilde{\mathcal{F}}^+_{i_0}} x_{i_0}$.

The result of Theorem 1 enables us to design a procedure that successively replaces some of the complementarity constraints with linear equations. This procedure will be described in the next section.

*4.2. On finding a solution of the BC-MLCP*

In this subsection, we describe the different steps of our proposed solution approach for the BC-MLCP. Our approach proceeds in two main steps. First, we fix some of the complementarity constraints to linear equations. Then, we solve an MILP that is equivalent to the original BC-MLCP with possibly a reduced number of complementarity constraints. A more detailed description of each step is given as follows.

1. ***Replacing the complementarity constraints:*** At this stage, we implement the procedure described in Algorithm 1, which is the result of Theorem 1. **Algorithm 1 starts with the first complementarity variable,** $i_0 = 1$**, and minimizes it over the relaxed set** $\tilde{\mathcal{F}}$**. Depending on the result of this minimization, if one of the conditions in Theorem 1 is satisfied, then the complementarity constraint** $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ **is accordingly replaced in the set** $\tilde{\mathcal{F}}$**. Otherwise the complementarity constraint is left unchanged. The algorithm then repeats these steps for each of the other complementarity variables in turn,** $i_0 = 2, \ldots, n_1$**. The detailed step-by-step procedure is given in Algorithm 1.**
It is important to note that fixing one complementarity constraint may affect the fixing of another complementarity constraint in the sense that the feasible region would have shrunken. This suggests that the order in which the variables are considered can affect the total number of complementarity constraints fixed. Therefore, in our implementation we have decided to repeat Algorithm 1 until it is no longer possible to fix any complementarity constraints.
At the end of this step, we define the following sets: $N_1^+ = \{i \in N_1 : (q_i + A_i x)x_i = 0 \text{ is replaced by } q_i + A_i x = 0\}$ and $N_1^0 = \{i \in N_1 : (q_i + A_i x)x_i = 0 \text{ is replaced by } x_i = 0\}$.

**Algorithm 1** : Procedure to reduce the number of complementarity constraints

```
Initialize $i_0 = 1$
while $i_0 \le n_1$ do
    Solve the LP $\min_{(x,y)\in\tilde{\mathcal{F}}} x_{i_0}$.
    if the above LP is infeasible, then
        Stop and exit (the BC-MLCP is infeasible).
    else
        Let $(x^*, y^*)$ be its optimal solution.
        if $(q_i + A_i x^*)x_i^* = 0$ for all $i \in N_1$ and $x_i^* \in \{0,1\}$ for all $i \in B$ then
            Stop and output $x^*$ as a solution of the BC-MLCP.
        else
            if $x_{i_0}^* > 0$ then
                Replace the constraint $q_{i_0}x_{i_0} + \sum_{j=1}^{n} A_{i_0}y_{i_0 j} = 0$
                with the equation $q_{i_0} + A_{i_0}x = 0$.
                (We can further set $x_{i_0} = 1$ if $i_0 \in B$.)
            else
                Solve the LP $\min_{(x,y)\in\tilde{\mathcal{F}}_{i_0}^{+}} x_{i_0}$.
                if this LP is infeasible then
                    Set $x_{i_0} = 0$.
                end if
            end if
        end if
    end if
    $i_0 = i_0 + 1$
end while
```

2. ***Solving the final MILP reformulation of the BC-MLCP:*** The procedure of replacing the complementarity constraints may end with either $N_1^+ \cup N_1^0 = N_1$ or $N_1^+ \cup N_1^0 \subsetneq N_1$.
If $N_1^+ \cup N_1^0 = N_1$ then all the complementarity constraints have been fixed and therefore the set $\mathcal{F}$ is equivalent to

$$\mathcal{F}^1 = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{cc} q_i + A_i x = 0 & \text{for } i \in N_1^+ \\ x_i = 0 & \text{for } i \in N_1^0 \\ q_i + A_i x = 0 & \text{for } i \in N \setminus N_1 \\ 0 \le x_i \le 1 & \text{for } i \in N \setminus B \\ x_i \in \{0,1\} & \text{for } i \in B \end{array} \right\}.$$

Hence, we find a solution to the BC-MLCP by solving an MILP over the set $\mathcal{F}^1$ with a zero objective function.
If $N_1^+ \cup N_1^0 \subsetneq N_1$, i.e., only some of the complementarity constraints have been fixed and therefore the set $\mathcal{F}$ is equivalent to

$$\mathcal{F}^2 = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{cc} q_i + A_i x \geq 0 & \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0) \\ (q_i + A_i x) x_i = 0 & \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0) \\ q_i + A_i x = 0 & \text{for } i \in N_1^+ \\ x_i = 0 & \text{for } i \in N_1^0 \\ q_i + A_i x = 0 & \text{for } i \in N \setminus N_1 \\ 0 \leq x_i \leq 1 & \text{for } i \in N \setminus B \\ x_i \in \{0,1\} & \text{for } i \in B \end{array} \right\}.$$

Note that finding a solution in the set $\mathcal{F}^2$ is again equivalent to finding a solution of a BC-MLCP problem, but with a smaller set of complementarity variables. In our approach, we follow the idea of Sherali et al [55] and define an MILP in the form of (3), which is equivalent to finding a solution in the set $\mathcal{F}^2$. This MILP is defined by:

$$
\begin{aligned}
\min_{x,v,w} \quad & q^T v + \sum_{i \in N_1 \setminus (N_1^+ \cup N_1^0)} \sum_{j \in N} A_{ij} w_{ij} && && \text{(4a)}\\
s.t. \quad & \sum_{j=1}^{n} A_{kj} w_{ij} + q_k v_i \geq 0 && \text{for } i,k \in N_1 \setminus (N_1^+ \cup N_1^0) && \text{(4b)}\\
& \sum_{j=1}^{n} A_{kj} x_j + q_k \geq \sum_{j=1}^{n} A_{kj} w_{ij} + q_k v_i && \text{for } i,k \in N_1 \setminus (N_1^+ \cup N_1^0) && \text{(4c)}\\
& q_i + A_i x = 0 && \text{for } i \in N_1^+ && \text{(4d)}\\
& x_i = 0 && \text{for } i \in N_1^0 && \text{(4e)}\\
& q_i + A_i x = 0 && \text{for } i \in N \setminus N_1 && \text{(4f)}\\
& x_j \in \{0,1\} && \text{for } j \in B && \text{(4g)}\\
& 0 \leq w_{ij} \leq 1 && \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0), j \in N && \text{(4h)}\\
& w_{ii} = x_i && \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0) && \text{(4i)}\\
& v_i \in \{0,1\} && \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0) && \text{(4j)}\\
& w_{ij} \geq 0 && \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0), j \in N && \text{(4k)}\\
& w_{ij} \leq x_j && \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0), j \in N && \text{(4l)}\\
& w_{ij} \leq v_i && \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0), j \in N && \text{(4m)}\\
& w_{ij} + 1 \geq x_j + v_i && \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0), j \in N. && \text{(4n)}
\end{aligned}
$$

In [55], the authors developed an enumerative branch-and-bound algorithm for solving the MILP (4). We use the MIP solver of `CPLEX` [35].

## 5. Computational experiments

We report computational results on a variety of BC-MLCP instances. Some of these instances come from well-known engineering applications. We have used instances that are important from both the theoretical and the practical point of view.

All our routines were coded in the C programming language and compiled with *gcc 4.6* [33], and the results were obtained by running our code on a laptop with processor at 2.13 GHz and 8 GB of RAM. We used `CPLEX` version 12.6 [35] in our experiments. All the linear programming problems encountered in Step 1 (replacing complementarity constraints) of our approach were solved with the LP solver of `CPLEX`, while the mixed-integer problems encountered in Step 2 (MILP reformulation) were solved with the MIP solver of `CPLEX`.

### *5.1. Experiments with random BC-MLCP instances*

In this Section, we test the efficiency of our proposed algorithm on randomly generated BC-MLCP instances. For this purpose, we experiment with three different sets of instances. In the first experiment, we investigate the effect that the number of binary variables has on the performance of the algorithm, but with a fixed number of variables. The second experiment tests the performance of our approach on BC-MLCP instances and highlights the contribution of this paper. Finally, the third experiment illustrates how the performance of the algorithm is affected by both the density of the matrix $A$ and the proportion of complementarity versus free variables.

#### *5.1.1. Generation of the instances*

All three sets of test instances were generated as follows. First we generate the matrix $A$ with entries drawn randomly from a uniform distribution on the interval $[-20, 20]$. Then we generate a vector $x$ with entries drawn randomly from a uniform distribution on the interval $[0, 20]$ for the complementarity variables, and on $[-20, 20]$ for the free variables. We uniformly choose a percentage of these entries and change their values to either 0 or 1. This represents the percentage of binary variables present in the problem. If all the non-binary entries of $x$ are non-zero, we force some of them to be zero. The vector $q$ is then calculated in the following manner. For $i \in \{1, \ldots, n_1\}$, if $x_i \neq 0$ then $q_i = -A_i x$, while if $x_i = 0$ then $q_i = -A_i x + \epsilon$, with $\epsilon$ being uniformly drawn from $[-5, 5]$, and finally for $i \in \{n_1 + 1, \ldots, n_1 + n_2\}$, $q_i = -A_i x$. It is important to note that this procedure is run independently from the algorithm for solving the instances. The idea is to be able to obtain a BC-MLCP that has at least one feasible solution. This procedure has been used previously for generating random instances of linear programs with equilibrium constraints, see e.g. [62].

The characteristics specific to the instances in each experiment will be given in the next three subsections. These characteristics refer to the percentage of binary variables, $\alpha$, the percentage of complementarity versus free variables, $\rho$, and the density of the matrix $A$, $\Delta$.

### *5.1.2. First experiment*

In this experiment, we wanted to assess whether the percentage ($\alpha$) of the binary variables present in the problem could affect the computational time of our approach. For this experiment, we generated 10 instances with $n = 200, n_1 = n_2 = 100$ (i.e., $\rho = 50\%$) for each value of $\alpha \in \{10, 20, 30, \dots, 100\}$ (i.e., we uniformly chose $\alpha\%$ of the $n$ variables to be binary). We also set the density (percentage of non-zero elements) of the matrix $A$ to $\Delta = 90\%$. We present the general overview of this experiment in Figure 1. **In each of these graphs, the x-axis corresponds to the value of $\alpha \in \{10, 20, 30, \dots, 100\}$, while the y-axis shows the CPU times (in seconds) required to solve the instances. In Figure 1a, the box plots show the distribution of the CPU times for all ten instances for each value of $\alpha$, while Figure 1b provides the average CPU time for the ten instances.**

**It can be seen in both graphs of Figure 1 that the CPU times of instances with a smaller percentage of binary variables are greater than the CPU times of instances with a higher percentage of binary variables. For example, it took up to 600 seconds to solve instances for which only 10% of the variables are required to be binary, whereas at most 45 seconds were required for instances where all the variables are binary. This suggests that the more binary variables there are in the BC-MLCP problem, the more efficient our proposed method is expected to be. The results of the next subsection also support this observation. Our explanation of this behaviour is that the relaxed set $\tilde{\mathcal{F}}$ becomes a tighter approximation of the original set $\mathcal{F}$ when there are more binary variables because the McCormick inequalities are equivalent to $y_{ij} = x_i x_j$ when both $x_i$ and $x_j$ are binary.**

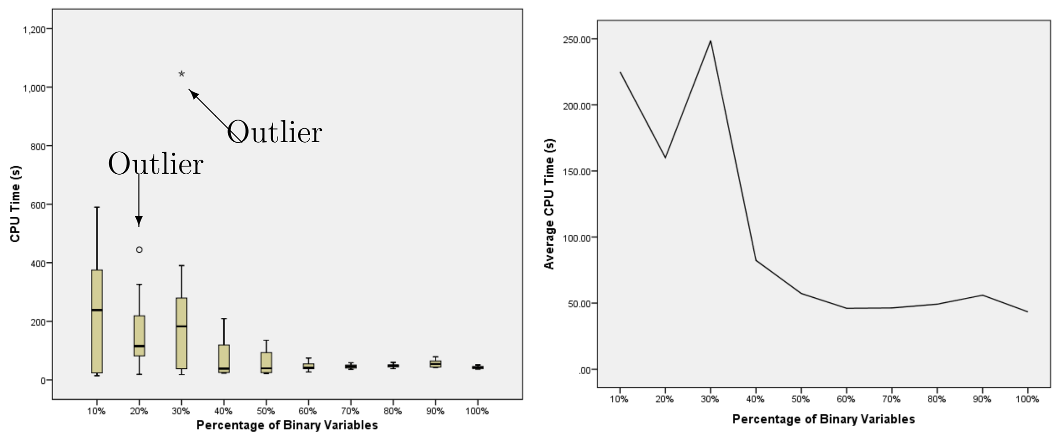
### *5.1.3. Second experiment*

Our second experiment tests the performance of our approach on BC-MLCP instances. For this experiment, we generated 10 instances for each value of $n \in \{20, 40, 60, \dots, 200\}$, $n_1 = n_2 = n/2$ (i.e., $\rho = 50\%$) and for each value of $\alpha \in \{20, 40, 60, 80\}$. Here we also assume that 90% of the entries in

Figure 1. CPU time Vs percentage of binary variables

(a) General tendency

(b) The averages




the matrix $A$ are non-zero entries. The results of this experiment are shown in Table 1. The size of each instance is given in column '$n$', the average number of complementarity constraints fixed by Algorithm 1 is reported in column 'Fixed'. Column '#S2' gives the number of instances, out of 10, for which the MILP reformulation phase (Step 2) of our approach was needed by the overall algorithm (Step 1 + Step 2) to solve the problem.

We report in Table 1 three sets of computational times (in seconds) needed to solve the instances. The first one (column 'S1') is the time taken to solve the problems when Step 1 alone is considered. The second one (column 'S2') is the time taken to solve the problem if the MILP (Step 2 alone) reformulation is solved directly with `CPLEX` without our proposed pre-processing phase (Step 1).The third one (column 'S1S2') is the computational time of implementing the overall proposed algorithm (Step 1 + Step 2). Finally, the column 'Big-M' reports the time taken by the Big-M method (with $M = 1,000$) as described in Section 3.3. These computational times are averages over 10 instances.

These results show that our approach solves BC-MLCP instances more efficiently than if we solve the MILP reformulation of the BC-MLCP directly, as per Section 3.2. More importantly, it can be seen that 'Step 1', which is the main novelty in this paper, plays a very significant role in the overall algorithm. In fact, it can be seen in Table 1 that the MILP (Step 2) alone is very time-consuming, whereas the novel procedure of replacing complementarity constraints with linear equations speeds up the overall algorithm significantly. This step alone is able to yield a solution to the BC-MLCP without needing the MILP phase. Indeed, it can be observed that out of 400 instances tested here, the MILP reformulation phase (Step 2) was required only once.

**At this point one may think that Step 1 is enough to solve the BC-MLCP problem. This is not true neither theoretically, nor experimentally, and we believe that these results are due to the structure of these instances. In Subsection 5.1.4 we conduct a third set of experiments using instances with more diverse structures, and the results show that when varying the structure of the problem in terms of number of complementarity constraints and density of the matrix, Step 2 is indeed needed to solve the corresponding BC-MLCP problems.**

On the other hand, while the Big-M approach is more efficient than our approach for the instances in the current section, we will show in Subsection 5.2 that the Big-M approach can fail on some simple instances of BC-MLCP for which our approach is able to find exact solutions. This adds to other drawbacks of the Big-M approach already mentioned in Section 3.

For the first two sets of experiments, one can notice that the running time of our approach gets relatively smaller when the percentage of binary variables present in the problem increases. This may be counter-intuitive for readers that are familiar with integer programming. Recalling that the McCormick inequalities are equivalent to $y_{ij} = x_i x_j$ when $x_i, x_j \in \{0, 1\}$, our explanation for this behavior is that, when there are more binary variables in the problem, replacing the quadratic identities $y_{ij} = x_i x_j$ with the McCormick inequalities yields a tighter relaxation of the original problem. Moreover, unlike in integer programming where one is looking for an optimal solution, in this problem we are only searching for a feasible solution.

#### *5.1.4. Third experiment*

We carried out a third set of experiments with randomly generated BC-MLCP instances to investigate how the density (number of non zero elements) of the matrix $A$ and the number of the complementarity variables could affect the performance of our approach. For this experiment, we generated 10 instances for each combination of $n \in \{20, 40, 60, \dots, 200\}, \Delta \in \{25\%, 50\%, 75\%\}$ and respectively 50%, 70% and 90% (so $\rho = 50, 70, 90$) of variables being complementarity variables, where $\Delta$ is the density of the matrix $A$ , and we fix $\alpha = 50$. This resulted in a total of 900 additional test instances for our approach. The results of this experiment are reported in Table 2. We notice that when we consider 50% of the variables to be complementarity variables, Algorithm 1 (Step 1) often solves the BC-MLCP problem, regardless of the density of the matrix. When the number of complementarity

Table 1. Results of randomly generated BC-MLCP instances

| | | | With 20% Binary | | | | | | | With 40% Binary | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Time (s) | | | | | | | Time (s) | | | |
| $n$ | Fixed | #S2 | S1 | S2 | S1S2 | Big-M | $n$ | Fixed | #S2 | S1 | S2 | S1S2 | Big-M |
| 20 | 2 | 1 | 0.03 | 0.11 | 0.08 | 0.04 | 20 | 1 | 0 | 0.19 | 0.06 | 0.20 | 0.02 |
| 40 | 2 | 0 | 0.39 | 2.69 | 0.41 | 0.5 | 40 | 1 | 0 | 0.57 | 2.15 | 0.58 | 0.04 |
| 60 | 4 | 0 | 0.69 | 15.28 | 0.76 | 0.14 | 60 | 1 | 0 | 3.02 | 3.43 | 3.05 | 0.07 |
| 80 | 3 | 0 | 5.57 | 77.64 | 5.93 | 0.17 | 80 | 2 | 0 | 4.07 | 24.28 | 4.35 | 0.08 |
| 100 | 7 | 0 | 16.31 | 541.56 | 18.04 | 0.25 | 100 | 1 | 0 | 3.72 | 48.67 | 3.81 | 0.13 |
| 120 | 5 | 0 | 21.29 | 1650.21 | 23.16 | 0.45 | 120 | 4 | 0 | 18.55 | 683.42 | 19.25 | 0.19 |
| 140 | 9 | 0 | 64.65 | 5786.78 | 67.46 | 0.4 | 140 | 1 | 0 | 9.95 | 740.94 | 10.15 | 0.21 |
| 160 | 8 | 0 | 117.88 | 24598.38 | 120.13 | 0.75 | 160 | 2 | 0 | 39.95 | 1411.77 | 40.21 | 0.33 |
| 180 | 10 | 0 | 175.34 | 25689.08 | 194.34 | 0.54 | 180 | 2 | 0 | 55.07 | 7356.48 | 55.23 | 0.42 |
| 200 | 4 | 0 | 103.82 | 32740.1 | 113.89 | 0.92 | 200 | 2 | 0 | 44.35 | 6419.04 | 46.26 | 0.55 |
| | | | With 20% Binary | | | | | | | With 40% Binary | | | |
| | | | Time (s) | | | | | | | Time (s) | | | |
| $n$ | Fixed | #S2 | S1 | S2 | S1S2 | Big-M | $n$ | Fixed | #S2 | S1 | S2 | S1S2 | Big-M |
| 20 | 1 | 0 | 0.08 | 0.05 | 0.09 | 0.02 | 20 | 0 | 0 | 0.04 | 0.02 | 0.04 | 0.01 |
| 40 | 0 | 0 | 0.42 | 0.54 | 0.43 | 0.05 | 40 | 0 | 0 | 0.32 | 0.62 | 0.33 | 0.04 |
| 60 | 1 | 0 | 0.97 | 5.69 | 0.97 | 0.06 | 60 | 1 | 0 | 1.63 | 3.85 | 1.74 | 0.07 |
| 80 | 0 | 0 | 0.74 | 14.86 | 0.77 | 0.06 | 80 | 0 | 0 | 1.62 | 12.97 | 1.67 | 0.06 |
| 100 | 2 | 0 | 1.08 | 18.17 | 1.14 | 0.11 | 100 | 1 | 0 | 2.42 | 14.15 | 2.54 | 0.1 |
| 120 | 1 | 0 | 2.81 | 40.78 | 2.81 | 0.21 | 120 | 1 | 0 | 2.95 | 25.98 | 3.11 | 0.14 |
| 140 | 0 | 0 | 3.34 | 44.31 | 3.62 | 0.26 | 140 | 1 | 0 | 4.49 | 111.36 | 4.73 | 0.25 |
| 160 | 0 | 0 | 4.29 | 167.73 | 4.34 | 0.32 | 160 | 1 | 0 | 5.4 | 210.59 | 5.63 | 0.25 |
| 180 | 1 | 0 | 6.63 | 174.44 | 6.85 | 0.43 | 180 | 1 | 0 | 9.74 | 469.26 | 10.02 | 0.34 |
| 200 | 0 | 0 | 9.97 | 1347.41 | 10.08 | 0.51 | 200 | 1 | 0 | 12.08 | 699.28 | 12.18 | 0.37 |

variables increases (70% and 90%), the MILP reformulation phase (Step 2) of our approach becomes necessary, but the number of complementarity constraints fixed increases, which helps the MIP solver to quickly find a solution when solving the MILP reformulation. We note that the computational time also becomes very large for the instances with higher percentages of complementarity.

Overall, this third set of computational experiments with randomly generated instances shows that our approach can effectively solve BC-MLCP instances without assuming any specific structure about the problem. The pre-processing phase (Step 1) of our approach is indeed very important in reducing the number of complementarity constraints. In most cases, this pre-processing phase alone is capable of solving the BC-MLCP.

### *5.2. Illustration of the limitations of the Big-M technique for BC-MLCP*

In this subsection we show the limitations of the Big-M technique for a class of instances of BC-MLCP related to the computational problems associated with the Hilbert matrix ($K$ defined below) as discussed in [47, Page 56].

Consider the following linear programming problem with $k$ decision vari-

Table 2. Results of randomly generated BC-MLCP instances for a combination of 50%, 70% and 90% of complementarity with various values of the density of the matrix $A$, $\Delta \in \{25\%, 50\%, 75\%\}$, $\alpha = 50$.

| | $\Delta = 25\%$ | | | | $\Delta = 50\%$ | | | | $\Delta = 75\%$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | n | Fixed | #S2 | Time | n | Fixed | #S2 | Time | n | Fixed | #S2 | Time |
| $\rho = 50\%$ | 20 | 1 | 0 | 0.00 | 20 | 1 | 0 | 0.00 | 20 | 1 | 0 | 0.01 |
| | 40 | 0 | 0 | 0.03 | 40 | 0 | 0 | 0.03 | 40 | 0 | 0 | 0.03 |
| | 60 | 1 | 0 | 0.24 | 60 | 1 | 0 | 0.40 | 60 | 0 | 0 | 0.16 |
| | 80 | 0 | 0 | 0.42 | 80 | 0 | 0 | 0.44 | 80 | 0 | 0 | 0.65 |
| | 100 | 0 | 0 | 0.87 | 100 | 0 | 0 | 0.83 | 100 | 0 | 0 | 1.17 |
| | 120 | 1 | 0 | 2.90 | 120 | 0 | 0 | 1.82 | 120 | 1 | 0 | 4.22 |
| | 140 | 0 | 0 | 2.54 | 140 | 0 | 0 | 2.01 | 140 | 0 | 0 | 2.50 |
| | 160 | 0 | 0 | 2.55 | 160 | 0 | 0 | 2.43 | 160 | 0 | 0 | 3.76 |
| | 180 | 0 | 0 | 4.94 | 180 | 0 | 0 | 4.30 | 180 | 0 | 0 | 4.96 |
| | 200 | 0 | 0 | 5.93 | 200 | 0 | 0 | 5.60 | 200 | 0 | 0 | 9.16 |
| | $\Delta = 25\%$ | | | | $\Delta = 50\%$ | | | | $\Delta = 75\%$ | | | |
| | n | Fixed | #S2 | Time | n | Fixed | #S2 | Time | n | Fixed | #S2 | Time |
| $\rho = 70\%$ | 20 | 2 | 1 | 0.01 | 20 | 2 | 1 | 0.03 | 20 | 2 | 0 | 0.01 |
| | 40 | 4 | 1 | 0.36 | 40 | 3 | 2 | 0.89 | 40 | 4 | 0 | 0.55 |
| | 60 | 5 | 0 | 2.42 | 60 | 8 | 2 | 16.98 | 60 | 9 | 0 | 5.03 |
| | 80 | 9 | 4 | 121.55 | 80 | 7 | 2 | 75.63 | 80 | 6 | 0 | 29.88 |
| | 100 | 11 | 1 | 129.33 | 100 | 6 | 0 | 16.85 | 100 | 8 | 0 | 158.63 |
| | 120 | 12 | 0 | 47.84 | 120 | 8 | 0 | 31.18 | 120 | 12 | 1 | 389.50 |
| | 140 | 11 | 0 | 68.59 | 140 | 11 | 0 | 157.34 | 140 | 13 | 1 | 822.56 |
| | 160 | 13 | 1 | 506.82 | 160 | 15 | 2 | 1471.39 | 160 | 14 | 2 | 1118.01 |
| | 180 | 14 | 2 | 1637.20 | 180 | 16 | 0 | 501.00 | 180 | 15 | 2 | 1598.83 |
| | 200 | 22 | 3 | 3190.70 | 200 | 15 | 1 | 1548.32 | 200 | 14 | 2 | 1430.00 |
| | $\Delta = 25\%$ | | | | $\Delta = 50\%$ | | | | $\Delta = 75\%$ | | | |
| | n | Fixed | #S2 | Time | n | Fixed | #S2 | Time | n | Fixed | #S2 | Time |
| $\rho = 90\%$ | 20 | 6 | 5 | 0.08 | 20 | 3 | 0 | 0.02 | 20 | 4 | 0 | 0.05 |
| | 40 | 6 | 7 | 21.36 | 40 | 2 | 2 | 0.21 | 40 | 3 | 10 | 68.38 |
| | 60 | 7 | 9 | 543.89 | 60 | 5 | 10 | 82.39 | 60 | 9 | 10 | 178.67 |
| | 80 | 10 | 9 | 2190.36 | 80 | 7 | 6 | 51.43 | 80 | 8 | 10 | 2885.34 |
| | 100 | 8 | 10 | 2917.76 | 100 | 5 | 10 | 1039.59 | 100 | 7 | 10 | 3285.53 |
| | 120 | 5 | 10 | 3782.44 | 120 | 3 | 10 | 2166.80 | 120 | 3 | 10 | 3584.69 |
| | 140 | 4 | 10 | 3543.03 | 140 | 4 | 10 | 3425.35 | 140 | 6 | 10 | 4891.38 |
| | 160 | 6 | 10 | 5524.26 | 160 | 5 | 10 | 2845.27 | 160 | 1 | 10 | 5371.31 |
| | 180 | 3 | 10 | 6314.93 | 180 | 4 | 10 | 2832.82 | 180 | 15 | 10 | 5892.84 |
| | 200 | 11 | 10 | 8869.49 | 200 | 7 | 10 | 2899.62 | 200 | 5 | 10 | 9974.47 |

ables:

$$\min_{y} \quad c^T y \tag{5a}$$

$$\mathbf{s.t.} \quad \mathbf{Ky \leq b} \tag{5b}$$

$$y \geq 0, \tag{5c}$$

where $c, K$ and $b$ are defined as follows:

$$K = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{k+2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{2k} \end{pmatrix},$$

$$b = (b_i, i = 1, \ldots, k)^T = \left( \sum_{j=1}^{k} \frac{1}{i+j} \right),$$

and

$$c = (c_j, j = 1, \ldots, k)^T = \left( \frac{2}{j+1} + \sum_{i=2}^{k} \frac{1}{j+i} \right).$$

This problem is known to have a unique optimal solution $y^* = (1, 1, \ldots, 1)$ and the dual problem also has a unique optimal solution $\pi^* = (2, 1, \ldots, 1)$ [47]. Furthermore, the results obtained when solving this problem using pivoting algorithms or a complementarity pivot algorithm through LCP formulation are very poor, especially when $k \geq 10$ (see [47]). The LCP formulation of such a problem can be regarded as a BC-MLCP where all the entries of $x$ are constrained to be binary except for the $(k+1)^{st}$ entry which is continuous.

In this experiment we solve the resulting BC-MLCP using the Big-M approach as introduced in Subsection 3.3 with the value of $M = 10$. The Big-M approach returns the problem as infeasible even for very small value of $k$ ($k \geq 3$). This is clearly because it is not capable of identifying a solution that satisfies the binary conditions of the variables.

While the infeasibility result from the Big-M approach suggests that the value of $M$ is too small, it is easy to check that $M = 10$ is a suitable value for this example. Relaxing the binary conditions, we obtained the solutions:

$$(y, \pi) = (0.997, 1.007, 0.994, 1.988, 1.043, 0.965) \text{ when } k = 3,$$

$$(y, \pi) = (0.985, 1.096, 0.823, 1.096, 1.979, 1.152, 0.696, 1.176) \text{ when } k = 4,$$

$$(y, \pi) = (0.95, 1.32, 0.66, 0.00, 3.05, 0.00, 1.94, 1.37, 0.58, 0.00, 3.09, 0.00) \text{ when } k = 6.$$

In contrast, our approach was capable of finding the exact solution for these instances for at least such small values of $k$, through the process of replacing complementarity constraints with linear equations of Algorithm 1. More specifically, in Algorithm 1 when we have $\min_{(x,y) \in \tilde{\mathcal{F}}} x_{i_0} > 0$ for a binary variable $x_{i_0}$, we set $x_{i_0} = 1$. This allowed our approach to find the exact solution to the above instances.

### 5.3. A two-node energy system

This example, taken from [29], presents a Nash-Cournot game between two energy producers. Each producer decides on their production level $q_p, p \in \{1,2\}$ to maximize their profit function $[a - b(q_1+q_2)]q_p - \big(\beta_p q_p^2 + \rho_p q_p\big)$ subject to their production capacity constraints $s_p q_{min} \leq q_p \leq s_p q_{max}$, where $s_p$ is a binary variable that is 1 if player $p$ decides to produce and 0 if not. Here the binary variable $s_p$ might for example relate to the on/off status for a power generation unit. If on, then the minimum and maximum production quantities are in force. If off, then both the upper and lower bounds are equal to zero. The BC-MLCP model associated with this example is defined as follows.

$$0 \leq -2q_p(b+\beta_p) + bq_{-p} - (a-\rho_p) + (\lambda_p^{max} - \lambda_p^{min}) \perp q_p \geq 0, \tag{6a}$$

$$0 \leq s_p q_{max} - q_p \perp \lambda_p^{max} \geq 0, \tag{6b}$$

$$0 \leq -s_p q_{min} + q_p \perp \lambda_p^{min} \geq 0, \tag{6c}$$

$$0 \leq 1 - s_p \perp \delta_p \geq 0, \tag{6d}$$

$$s_p \in \{0,1\}, \tag{6e}$$

for all $p \in \{1,2\}$. The parameters are $a = 9, b = 1, \beta_1 = \beta_2 = 1, \rho_1 = 1, \rho_2 = 3, q_{min} = 1.5$ and $q_{max} = 4$.

The results are reported in Table 3. They are similar to the ones found by Gabriel et al. [29]. This does not necessarily suggest that there is a unique solution to this problem. Note that all the complementarity constraints are satisfied as well as the binary requirement on the variables $s_1$ and $s_2$. For this instance, Algorithm 1 was able to provide a solution that satisfies both requirements without the need of the MILP reformulation phase (Step 2) of our approach.

Table 3. Output of the two-node energy system

| Param. | $q_1$ | $q_2$ | $s_1$ | $s_2$ | $\lambda_1^{max}$ | $\lambda_2^{max}$ | $\lambda_1^{min}$ | $\lambda_2^{min}$ |
|---|---|---|---|---|---|---|---|---|
| Values | 1.62 | 1.5 | 1 | 1 | 0.0 | 0.0 | 0.0 | 1.62 |

### 5.4. A market-clearing problem as BC-MLCP

In order to test our approach on more practical BC-MLCP instances, we consider the power market-clearing problem presented in Section 3 of [28]. This is a market-clearing problem of a multi-period power network, which features 6 (production/consumption) nodes, 8 producers and 4 demand blocks. The model corresponds to a multi-period auction in which producers submit production offers consisting of energy blocks and their corresponding selling prices as well as start-up and shut-down costs. In addition, consumers submit consumption bids consisting of energy blocks and their corresponding buying prices. This model clears the market by maximizing the social welfare which is computed using the producers offers and the consumers bids (i.e., the "declared" social welfare). We present the model in Appendix A, and refer the reader to [28] for a full description of this problem and the definition of its parameters. The variables of this problems are:

- $P^G_{tib}$: the power produced by block $b$ of unit $i$ in period $t$.
- $P^D_{tjk}$: the power consumed by block $k$ of demand $j$ in period $t$.
- $\mu^{Gmax}_{tib}$: the Lagrangian multiplier associated with the bound of the production for block $b$ of unit $i$ in period $t$.
- $\mu^{Dmax}_{tjk}$: the Lagrangian multiplier associated with the bound of the demand for block $k$ of unit $j$ in period $t$.
- $\mu^{\min}_{ti}$: the Lagrangian multiplier associated with the minimum power output of unit $i$ in period $t$.
- $\nu^{min}_{tnm}$ and $\nu^{max}_{tnm}$: the Lagrangian multipliers associated with the limits of the transmission capacity of the line relating nodes $n$ and $m$ in period $t$.
- $\delta_{tn}$: the voltage angle of node $n$ in period $t$.
- $\xi^{\max}_{tn}$: the Lagrangian multiplier associated with $\delta_{tn}$.
- $C^U_{ti}$ and $C^D_{ti}$: are respectively the non-negative start-up and shut-down cost of unit $i$ in period $t$. Their corresponding Lagrangian multipliers are $\eta^U_{ti}$ and $\eta^D_{ti}$, respectively.
- $v_{ti}$: the binary variable describing the on/off status of unit $i$ in period $t$. Its corresponding Lagrangian multiplier is $\beta^{\max}_{ti}$.
- $\lambda_{tn}$: the Lagrangian multiplier associated with the equation that enforces the power balance at node $n$ in period $t$.
- $\xi^1_{t(n=1)}$: the Lagrangian multiplier associated with the equation that imposes the reference node.

We report the values of the variables related to each producer in Table 4. The values of variables related to each demand block and the values of nodes specific variables are given in Table 5 and in Table 6, respectively. In addition to the values reported in these tables we also had $\nu_{tnm}^{min} = 0, \nu_{tnm}^{max} = 0$ for all $t, n, m$, and $\xi_{t(n=1)}^{1} = 0$ for all $t$. Some of these values match with those reported in [28]. For this problem, the MILP reformulation phase (Step 2) of our approach was required in order to find a solution. However, Algorithm 1 was able to replace up to 99 (out of 248) complementarity constraints with linear equations.

Table 4. Producers variables

| Producers | $P_{tib}^{G}$ | | $\mu_{tib}^{G}$ | | $\mu_{ti}^{min}$ | | $\eta_{ti}^{U}$ | | $C_{ti}^{U}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | t=1 | t=2 | t=1 | t=2 | t=1 | t=2 | t=1 | t=2 | t=1 | t=2 |
| $i = 1$ | 0 | 0 | 2 | 0 | 0 | 10 | 0 | 0 | 0 | 0 |
| $i = 2$ | 0 | 0 | 4 | 0 | 0 | 8 | 1 | 0 | 0 | 0 |
| $i = 3$ | 50 | 25 | 6 | 0 | 0 | 6 | 0.8 | 0 | 0 | 0 |
| $i = 4$ | 50 | 25 | 8 | 0 | 0 | 4 | 0 | 0 | 0 | 0 |
| $i = 5$ | 50 | 25 | 10 | 0 | 0 | 2 | 0 | 0 | 0 | 0 |
| $i = 6$ | 50 | 25 | 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $i = 7$ | 50 | 50 | 14 | 2 | 0 | 0 | 1 | 0 | 350 | 0 |
| $i = 8$ | 50 | 50 | 16 | 4 | 0 | 0 | 1 | 0.4 | 500 | 0 |
| Producers | $\eta_{ti}^{D}$ | | $C_{ti}^{D}$ | | $\beta_{ti}^{\max}$ | | $v_{ti}$ | | | |
| | t=1 | t=2 | t=1 | t=2 | t=1 | t=2 | t=1 | t=2 | | |
| $i = 1$ | 0 | 0.2 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| $i = 2$ | 0 | 0.57 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| $i = 3$ | 0 | 0.5 | 0 | 0 | 0 | 0 | 1 | 1 | | |
| $i = 4$ | 0 | 0.4 | 0 | 0 | 300 | 0 | 1 | 1 | | |
| $i = 5$ | 0 | 0.22 | 0 | 0 | 450 | 0 | 1 | 1 | | |
| $i = 6$ | 0 | 0 | 0 | 0 | 600 | 0 | 1 | 1 | | |
| $i = 7$ | 0 | 0 | 0 | 0 | 350 | 100 | 1 | 1 | | |
| $i = 8$ | 0 | 0 | 0 | 0 | 500 | 0 | 1 | 1 | | |

Table 5. Demands variables

| Demands | $P_{tjk}^{D}$ | | $\mu_{tjk}^{D}$ | |
|---|---|---|---|---|
| | t=1 | t=2 | t=1 | t=2 |
| $j = 1$ | 0 | 50 | 0 | 6 |
| $j = 2$ | 100 | 50 | 0 | 6 |
| $j = 3$ | 100 | 50 | 0 | 7 |
| $j = 4$ | 100 | 50 | 1 | 7 |

Table 6. Nodes variables

| Nodes | $\delta_{tn}$ | | $\xi_{tn}^{\max}$ | | $\lambda_{tn}$ | |
|---|---|---|---|---|---|---|
| | t=1 | t=2 | t=1 | t=2 | t=1 | t=2 |
| $n = 1$ | 0 | 0 | 0 | 0 | 26 | 14 |
| $n = 2$ | 0.002 | -0.005 | 0 | 0 | 26 | 14 |
| $n = 3$ | -0.0017 | 0.005 | 0 | 0 | 26 | 14 |
| $n = 4$ | 0.1067 | 0.03 | 0 | 0 | 26 | 14 |
| $n = 5$ | 0.125 | 0.025 | 0 | 0 | 26 | 14 |
| $n = 6$ | 0.0933 | 0.02 | 0 | 0 | 26 | 14 |

## 6. Conclusions

We have presented a novel exact solution approach for the BC-MLCP. Our approach starts with a sophisticated pre-processing which considers the RLT relaxation of the complementarity constraints and enables us to replace some of the complementarity constraints with linear equations by solving a series of LPs. This pre-processing is then followed by the solution of an equivalent MILP reformulation of the BC-MLCP with a reduced number of complementarity constraints, which ensures that our approach always finds a solution that satisfies all the complementarity constraints as well as all the integrality conditions. Our computational results on a variety of both practical and theoretical examples show that the pre-processing often suffices to find a solution. Some of the comparisons provided also show that solving the MILP reformulation of the BC-MLCP without our proposed pre-processing can be significantly more expensive computationally. This suggests that our pre-processing can be used to improve the efficiency of other BC-MLCP algorithms.

For future research, we could incorporate cutting planes in the second phase (Algorithm 1) of our approach to strengthen the RLT relaxation. Several cutting planes have been proposed in the literature for this purpose, see for example [6, 16]. This might enable Algorithm 1 to find a solution in fewer iterations.

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## Appendix A. Market-clearing model

$$0 \leq P^G_{tib} \perp \lambda^G_{tib} - \lambda_{tn} + \mu^{Gmax}_{tib} - \mu^{min}_{ti} \geq 0, \qquad \forall t, \forall i \in \Psi_n, \forall b \tag{A.1a}$$

$$0 \leq P^D_{tjk} \perp -\lambda^D_{tjk} + \lambda_{tn} + \mu^{Dmax}_{tjk} \geq 0, \qquad \forall t, \forall j \in \Psi_n, \forall k \tag{A.1b}$$

$$0 \leq \mu^{Gmax}_{tib} \perp \nu_{ti} P^{Gmax}_{tib} - P^G_{tib} \geq 0, \qquad \forall t, \forall j, \forall k \tag{A.1c}$$

$$0 \leq \mu^{Dmax}_{tjk} \perp P^{Dmax}_{tjk} - P^D_{tjk} \geq 0, \qquad \forall t, \forall i, \forall b \tag{A.1d}$$

$$0 \leq \mu^{min}_{ti} \perp \sum_b P^G_{tib} - \nu_{ti} P^{Gmin}_{ti} \geq 0, \qquad \forall t, \forall i \tag{A.1e}$$

$$0 \leq \upsilon^{min}_{tnm} \perp B_{nm}(\delta_{tn} - \delta_{tm}) + P^{max}_{nm} \geq 0, \qquad \forall t, \forall n, \forall m \in \Theta_n \tag{A.1f}$$

$$0 \leq \upsilon^{max}_{tnm} \perp P^{max}_{nm} - B_{nm}(\delta_{tn} - \delta_{tm}) \geq 0, \qquad \forall t, \forall n, \forall m \in \Theta_n \tag{A.1g}$$

$$\begin{aligned} 0 \leq \delta_{tn} + \pi \perp & \sum_{m \in \Theta_n} B_{nm}(\lambda_{tn} - \lambda_{tm}) \\ & + \sum_{m \in \Theta_n} B_{nm}(\upsilon^{min}_{tmn} - \upsilon^{min}_{tnm}) + \xi^{max}_{tn} + (\xi^1_{t(n=1)}) \geq 0, \qquad \forall t, \forall n \end{aligned} \tag{A.1h}$$

$$0 \leq \pi - \delta_{tn} \perp \xi^{max}_{tn} \geq 0, \qquad \forall t, \forall n \tag{A.1i}$$

$$0 \leq \eta^U_{ti} \perp C^U_{ti} - (\upsilon_{ti} - \upsilon_{(t-1)i}) K^U_i \geq 0, \qquad \forall t, \forall i \tag{A.1j}$$

$$0 \leq 1 - \eta^U_{ti} \perp C^U_{ti} \geq 0, \qquad \forall t, \forall i \tag{A.1k}$$

$$0 \leq \eta^D_{ti} \perp C^D_{ti} - (\upsilon_{(t-1)i} - \upsilon_{ti}) K^D_i \geq 0, \qquad \forall t, \forall i \tag{A.1l}$$

$$0 \leq 1 - \eta^D_{ti} \perp C^D_{ti} \geq 0, \qquad \forall t, \forall i \tag{A.1m}$$

$$\begin{aligned} 0 \leq \upsilon_{ti} \perp & \mu^{min}_{ti} P^{Gmin}_{ti} - \sum_b \mu^{Gmax}_{tib} P^{Gmax}_{tib} + K^U_i(\eta^U_{ti} - \eta^U_{(t+1)i}) \\ & - K^D_i(\eta^D_{ti} - \eta^D_{(t+1)i}) + \beta^{max}_{ti} \geq 0, \qquad \forall t < T, \forall i \end{aligned} \tag{A.1n}$$

$$\begin{aligned} 0 \leq \upsilon_{ti} \perp & \mu^{min}_{ti} P^{Gmin}_{ti} - \sum_b \mu^{Gmax}_{tib} P^{Gmax}_{tib} + K^U_i \eta^U_{ti} \\ & - K^D_i \eta^D_{ti} + \beta^{max}_{ti} \geq 0, \qquad \forall t = T, \forall i \end{aligned} \tag{A.1o}$$

$$0 \leq 1 - \upsilon_{ti} \perp \beta^{max}_{ti} \geq 0, \qquad \forall t, \forall i \tag{A.1p}$$

$$\sum_{(i \in \Psi_n) b} P^G_{tib} - \sum_{(j \in \Psi_n) k} P^D_{tjk} = \sum_{m \in \Theta_n} B_{nm}(\delta_{tn} - \delta_{tm}), \qquad \forall t, \forall n \tag{A.1q}$$

$$\delta_{tn} = 0, \qquad \forall t, n = 1 \tag{A.1r}$$

## Appendix B. Proof of Lemma 1

*Proof.* From the lemma's premise, there exists finite bounds such that $0 \leq z_i^1 \leq u_i^1$ for $i \in N_1 \setminus B$ and $l_i^2 \leq z_i^2 \leq u_i^2$ for $i \in N_2 \setminus B$. Thus, we can form a new variables $s_i^1 = \frac{z_i^1}{u_i^1}$ and $s_i^2 = \frac{z_i^2 - l_i^2}{u_i^2 - l_i^2}$ with $0 \leq s_i^1, s_i^2 \leq 1$. The original LCP $(q, A)$ is given as

$$\begin{array}{rcl} 0 & \leq & q^1 + A^{11}z^1 + A^{12}z^2 \perp z^1 \geq 0 \\ 0 & = & q^2 + A^{21}z^1 + A^{22}z^2, z^2 \text{ free} \end{array}$$

where $z^1, z^2$ are respectively, the vector of nonnegative and free variables and where $q = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix}, A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}$, conformal with $z^1, z^2$. Define the positive diagonal matrices $D^1 = \text{diag}\,(u^1)$ and $D^2 = \text{diag}\,(u^2 - l^2)$ whose diagonals are the $u_i^1$ and $u_i^2 - l_i^2$ values for $z^1$ and $z^2$, respectively. Clearly, in light of the positive diagonals of $D^1, s_i^1 = 0 \Leftrightarrow z_i^1 = 0$ and $s_i^1 > 0 \Leftrightarrow z_i^1 > 0$. Note that $D^2$ also has positive diagonals. We do have the following:

$$\begin{array}{rcl} q^1 + A^{11}z^1 + A^{12}z^2 & = & q^1 + A^{11}D^1s^1 + A^{12}D^2s^2 + A^{12}l^2 \\ & = & \left(q^1 + A^{12}l^2\right) + A^{11}D^1s^1 + A^{12}D^2s^2 \\ & = & \widetilde{q}^1 + \widetilde{A}^{11}s^1 + \widetilde{A}^{12}s^2, \end{array}$$

where $\widetilde{q}^1 = q^1 + A^{12}l^2, \widetilde{A}^{11} = A^{11}D^1$ and $\widetilde{A}^{12} = A^{12}D^2$.

Analogously, $0 = q^2 + A^{21}z^1 + A^{22}z^2 \Leftrightarrow 0 = \widetilde{q}^2 + \widetilde{A}^{21}s^1 + \widetilde{A}^{22}s^2$, with $\widetilde{q}^2 = q^2 + A^{22}l^2, \widetilde{A}^{21} = A^{21}D^1$ and $\widetilde{A}^{22} = A^{22}D^2$, so that the solution set of LCP$(q, A)$ maps one-to-one to the solution set for LCP$(\widetilde{q}, \widetilde{A})$, where $\widetilde{q} = \begin{pmatrix} \widetilde{q}^1 \\ \widetilde{q}^2 \end{pmatrix}$, and $\widetilde{A} = \begin{pmatrix} \widetilde{A}^{11} & \widetilde{A}^{12} \\ \widetilde{A}^{21} & \widetilde{A}^{22} \end{pmatrix}$. Thus, without loss of generality it can be assumed that $z_i \in [0, 1]$ for all $i \in N \setminus B$.

$\square$